Linear Operators Preserving the (p,q)-Numerical Range

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Dedicated to our respected teacher Dr. Yik-Hoi Au-Yeung.

ABSTRACT

Let $\mathbf{C}_{n\times n}$ and $\mathscr{H}_n$ denote respectively the space of $n\times n$ complex matrices and the real space of $n\times n$ hermitian matrices. Let p,q,n be positive integers such that $p\leqslant q\leqslant n$. For $A\in\mathbf{C}_{n\times n}$, the (p,q)-numerical range of A is the set

$$W_{p,q}(A)=\left\{\operatorname{tr}C_p\left(J_qUAU^*\right):U\text{ unitary}\right\},$$

where $C_p(X)$ is the pth compound matrix of X, and J_q is the matrix $I_q\oplus O_{n-q}$. Let $\mathscr{S}$

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denote $\mathcal{H}_n$ or $\mathbf{C}_{n\times n}$. The problem of determining all linear operators $T:\mathcal{S}\to\mathcal{S}$ such that

$$W_{p,q}(T(A)) = W_{p,q}(A) \qquad \text{for all} \quad A\in\mathcal{S}$$

is treated in this paper.

1. INTRODUCTION

Let $\mathbf{C}_{m\times n}$, $\mathcal{H}_n$, and $\mathcal{U}_n$ denote respectively the space of $m\times n$ complex matrices, the real linear space of $n\times n$ hermitian matrices, and the group of unitary matrices in $\mathbf{C}_{n\times n}$. For $A\in\mathbf{C}_{n\times n}$, we denote the transpose, the trace, and the pth compound matrix of A respectively by A^t, $\operatorname{tr} A$, and $C_p(A)$. The $p\times p$ identity matrix and the $q\times q$ zero matrix will be denoted respectively by I_p and O_q. We also use J_q to denote the $n\times n$ matrix $I_q\oplus O_{n-q}$ for a fixed $n\geqslant q$.

Let p,q,n be integers such that $1\leqslant p\leqslant q\leqslant n$. For $A\in\mathbf{C}_{n\times n}$, the (p,q)*-numerical range* of A is defined to be the set

$$W_{p,q}(A) = \left\{\operatorname{tr} C_p(J_q UAU^*): U\in\mathcal{U}_n\right\}.$$

A matrix of the form UAU^*, where $U\in\mathcal{U}_n$, will be called a *unitary transform* of A. It is not difficult to see that the set $W_{p,q}(A)$ consists of the values of the pth elementary symmetric function E_p at the eigenvalues of a $q\times q$ principal submatrix of a unitary transform of A. So, in particular, if the eigenvalues of A are $\lambda_1,\dots,\lambda_n$, then for any integers $1\leqslant i_1<\cdots<i_q\leqslant n$, $E_p(\lambda_{i_1},\dots,\lambda_{i_q})\in W_{p,q}(A)$.

As can be seen, the concept of (p,q)-numerical range covers as special cases the concepts of the classical numerical range, the q-numerical range, the qth decomposable numerical range, the trace, the determinant, and the pth elementary symmetric function of the eigenvalues of a matrix. There has been a great deal of interest in determining all linear operators which preserve one of the above invariants associated with a matrix. As suggested by Tin-Yau Tam a couple of years ago, it is natural to ask the question of determining all linear operators which preserve the (p,q)-numerical range. To be more specific, let $\mathcal{S}$ denote the real space $\mathcal{H}_n$ or the complex space $\mathbf{C}_{n\times n}$. Our problem is to determine all linear operators $T:\mathcal{S}\to\mathcal{S}$ such that

$$W_{p,q}(T(A)) = W_{p,q}(A) \qquad \text{for all} \quad A\in\mathcal{S}.$$

TABLE 1

Conditions on (p,q)	Specializations of $W_{p,q}(A)$	References	
		$\mathscr{H}_n$	$\mathbf{C}_{n\times n}$
$1=p=q<n$	$W(A)$, classical numerical range	[8]	[19]
$1=p<q=n$	$\{\operatorname{tr} A\}$	—	[7]
$1=p<q<n$	$W_q(A)$, q-numerical range	[9]	[20, 9]
$1<p=q=n$	$\{\det A\}$	—	[3]
$1<p=q<n$	$\hat{W}_q(A)$, qth decomposable numerical range	[22]	[13, 23]
$1<p<q=n$	$\{E_p(\lambda(A))\}$	—	[16, 1]

For convenience, we summarize in Table 1 the references for characterization theorems obtained so far on this problem. [We have excluded the trivial case when $1=p=q=n$. We have also used $E_p(\lambda(A))$ to denote $E_p(\lambda_1,\dots,\lambda_n)$, where $\lambda_1,\dots,\lambda_n$ are the eigenvalues of A.]

One remark is in order concerning the delicate case when $p<q=n$. Following Marcus and Purves [16], we call a linear operator $T:\mathscr{S}\to\mathscr{S}$ a *direct product* if there exists a scalar ξ and fixed U and V in $\mathbf{C}_{n\times n}$ such that

$$T(A)=\xi UAV \qquad \text{for all} \quad A\in\mathscr{S}$$

or

$$T(A)=\xi UA^tV \qquad \text{for all} \quad A\in\mathscr{S}.$$

Marcus and Purves [16] and Beasley [1] have completely characterized those linear operators preserving the (p,q)-numerical range for $\mathscr{S}=\mathbf{C}_{n\times n}$ and $3\leqslant p<q=n$. As pointed out in [16], for the case $p=1,2$, there are no nice results; not every linear operator which preserves the (p,q)-numerical range is a direct product. Borrowing their counterexamples, we easily see that the same remark also holds for the hermitian case.

The main purpose of this paper is to prove the following two results which cover the unknown cases of the problem of (p,q)-numerical range preservers.

THEOREM 1. *Let p,q,n be integers such that $2 \leqslant p \leqslant q \leqslant n$ but not $2 = p < q = n$. A linear operator $T: \mathscr{H}_n \to \mathscr{H}_n$ satisfies*

$$W_{p,q}(T(A)) = W_{p,q}(A) \qquad \textit{for all} \quad A \in \mathscr{H}_n$$

if and only if there exist $U \in \mathbf{C}_{n\times n}$ and a real number ξ with $\xi^p = 1$ such that

$$T(A) = \xi UAU^* \qquad \textit{for all} \quad A \in \mathscr{H}_n$$

or

$$T(A) = \xi UA^tU^* \qquad \textit{for all} \quad A \in \mathscr{H}_n,$$

where U satisfies $|\det U| = 1$ if $p = q = n$, and is unitary if $2 \leqslant p \leqslant q < n$ or $2 < p < q = n$.

Marcus and Moyls [15, Theorems 2 and 5] have characterized those linear operators $T: \mathbf{C}_{n\times n} \to \mathbf{C}_{n\times n}$ which preserve the determinant of hermitian matrices. Using their result, as suggested by Marcus, one can deduce the particular case when $p = q = n$ of Theorem 1.

THEOREM 2. *Let p,q,n be integers such that $2 \leqslant p \leqslant q < n$. A linear operator $T: \mathbf{C}_{n\times n} \to \mathbf{C}_{n\times n}$ satisfies*

$$W_{p,q}(T(A)) = W_{p,q}(A) \qquad \textit{for all} \quad A \in \mathbf{C}_{n\times n}$$

if and only if there exist $U \in \mathscr{U}_n$ and a complex number ξ with $\xi^p = 1$ such that

$$T(A) = \xi UAU^* \qquad \textit{for all} \quad A \in \mathbf{C}_{n\times n}$$

or

$$T(A) = \xi UA^tU^* \qquad \textit{for all} \quad A \in \mathbf{C}_{n\times n}.$$

2. SOME LEMMAS

The following lemma will be used several times in this paper. As pointed out by D. Ž. Djoković, it follows as a corollary of Pólya and Szegö [21, Part 5, Problem 49]. Here we give a short direct proof for the sake of completeness.

LEMMA 1. *Let $\lambda_1,\dots,\lambda_n$ be real numbers, not necessarily distinct. If there exists an integer p, $2 \leqslant p \leqslant n$, such that $E_p(\lambda_1,\dots,\lambda_n) = E_{p-1}(\lambda_1,\dots,\lambda_n) = 0$, then at most $p-2$ λ_i's are nonzero.*

Proof. Suppose we have exactly k nonzero λ_i's, say $\lambda_1,\dots,\lambda_k$. Assume to the contrary that $k \geqslant p-1$. If $k=p$, then $0 = E_p(\lambda_1,\dots,\lambda_n) = \prod_{i=1}^{p}\lambda_i \neq 0$, which is impossible. Similarly, the case $k = p-1$ cannot occur. So, henceforth, we assume that $k > p$. The given conditions on the λ_i's can be reduced to $E_p(\lambda_1,\dots,\lambda_k) = E_{p-1}(\lambda_1,\dots,\lambda_k) = 0$. Define

$$f(x) = \prod_{i=1}^{k}(x-\lambda_i).$$

In view of

$$\sum_{j=0}^{k}\frac{f^{(j)}(0)}{j!}x^j = f(x)$$

$$= x^k + \sum_{i=1}^{k}(-1)^i E_i(\lambda_1,\dots,\lambda_k)x^{k-i},$$

where $f^{(j)}$ denotes the jth derivative of f for $j \geqslant 1$ and $f^{(0)} = f$, we have, $f^{(k-p)}(0) = f^{(k-p+1)}(0) = 0$. In other words, $f^{(k-p)}$ has a repeated root at 0. On the other hand, since the roots of f are real, it follows from Rolle's theorem that for $i = 1,\dots,k-1$, the roots of $f^{(i)}$ are also real and interlace those of $f^{(i-1)}$. Hence 0 cannot be a repeated root of $f^{(k-p)}$. Thus our assumption at the beginning of the proof is wrong. ∎

LEMMA 2. *Let p,q,n be integers satisfying $2 \leqslant p \leqslant q \leqslant n$ but not $2 = p < q = n$. Then for any nonzero matrix $A \in \mathscr{H}_n$,* rank $A = 1$ *if and only if for all $B \in \mathscr{H}_n$, $U \in \mathscr{U}_n$,* $\operatorname{tr} C_p(J_qU(xA+B)U^*)$, *considered as a polynomial in x, is of degree at most one.*

(Cf. Marcus and Purves [16, Lemmas 3.2, 3.3] and Beasley [1, Lemma 2.2].)

Proof. "Only if": Let $A \in \mathscr{H}_n$ with rank $A = 1$. Consider any $U \in \mathscr{U}_n$ and $B \in \mathscr{H}_n$. Rewrite $J_qU(xA+B)U^*$ as $xA_1 + B_1$, where $A_1 = J_qUAU^*$ is of rank $\leqslant 1$ and $B_1 = J_qUBU^*$. Let $J = P^{-1}A_1P$ be the Jordan form of A_1. Then

J has at most one nonzero entry, and hence each entry of $C_p(xJ + P^{-1}B_1P)$ is a polynomial in x of degree at most one. Thus

$$\operatorname{tr} C_p\big(J_q U(xA + B)U^*\big) = \operatorname{tr} C_p\big(xPJP^{-1} + B_1\big) = \operatorname{tr} C_p\big(xJ + P^{-1}B_1P\big)$$

is a polynomial of degree at most one.

"If": Suppose $A \in \mathscr{H}_n$ with rank $A = k \geqslant 2$. We may assume that $A = \operatorname{diag}(a_1, \dots, a_n)$, where $a_1 \geqslant \cdots \geqslant a_k$ are all the nonzero eigenvalues of A. If $k \leqslant p$, take $B = O_k \oplus I_{p-k} \oplus O_{n-p}$. Then $\operatorname{tr} C_p(J_q(xA + B)) = (\prod_{i=1}^k a_i)x^k$ is a polynomial of degree $k \geqslant 2$.

Henceforth we assume that $k > p$. First, we consider the case $3 \leqslant p$. Take $B = J_t$, where $t = \min\{k, q\}$ $(\geqslant p)$. Then

$$\begin{aligned}
&\operatorname{tr} C_p\big(J_q(xA + B)\big)\\
&\quad = E_p(xa_1 + 1, \dots, xa_t + 1)\\
&\quad = \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant t} \big(xa_{i_1} + 1\big) \cdots \big(xa_{i_p} + 1\big)\\
&\quad = E_p(a_1, \dots, a_t)x^p + (t - p + 1)E_{p-1}(a_1, \dots, a_t)x^{p-1} + \cdots .
\end{aligned}$$

Since $a_1, \dots, a_t$ are all nonzero real numbers, by Lemma 1, $E_p(a_1, \dots, a_t)$ and $E_{p-1}(a_1, \dots, a_t)$ cannot be both zero. As $p \geqslant 3$, this shows that $\operatorname{tr} C_p(J_q(xA + B))$ is of degree $\geqslant 2$.

Now we consider the case $p = 2$. When $2 = p = q \leqslant n$, or A is a scalar matrix, clearly $\operatorname{tr} C_p(J_q(xA))$ is of degree 2. So we may assume $2 = p < q < n$ and A is a nonscalar matrix. Note that we have $a_1 \neq a_n$. Thus for any number λ lying between a_1 and a_n, we can find $V \in \mathscr{U}_2$ such that the matrix VDV^* has diagonal entries λ and $a_1 + a_n - \lambda$, where $D = \operatorname{diag}(a_1, a_n)$. Let $U \in \mathscr{U}_n$ be obtained from the identity matrix by replacing the $(1,1), (1,n), (n,1), (n,n)$ entries with the $(1,1), (1,2), (2,1), (2,2)$ entries of V. Then

$$\operatorname{tr} C_2\big(J_q(xUAU^*)\big) = E_2(\lambda, a_2, \dots, a_t)x^2,$$

where $t = \min\{k, q\} > 2$. Since

$$E_2(\lambda, a_2, \dots, a_t) = E_2(a_2, \dots, a_t) + \lambda E_1(a_2, \dots, a_t),$$

by Lemma 1 again, we can find some λ lying between a_1 and a_n such that $E_2(\lambda, a_2, \dots, a_t) \neq 0$. Hence $\operatorname{tr} C_2(J_q(xUAU^*))$ is of degree 2. The proof is complete. ■

LEMMA 3. *Let p, q, n be integers satisfying $2 \leqslant p \leqslant q < n$. Then for any $A \in \mathbf{C}_{n\times n}$, A is hermitian if and only if*

$$W_{p,q}(A + rI) \subseteq \mathbf{R} \qquad \textit{for all} \quad r \in \mathbf{R}.$$

Proof. The "only if" part is obvious. To prove the "if" part, suppose that $A \in \mathbf{C}_{n\times n}$ satisfies

$$W_{p,q}(A + rI) \subseteq \mathbf{R} \qquad \text{for all} \quad r \in \mathbf{R}.$$

Consider the $q \times q$ leading principle submatrix A_1 of an arbitrary unitary transform of A. Let the eigenvalues of A_1 be $\mu_1, \dots, \mu_q$. Then

$$\operatorname{tr} C_p(A_1 + rI_q) = \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant q} (\mu_{i_1} + r)\dots(\mu_{i_p} + r)$$

$$= \binom{q}{p} r^p + \binom{q-1}{p-1} E_1(\mu_1, \dots, \mu_q) r^{p-1} + \cdots$$

is a real number for all $r \in \mathbf{R}$, since $\operatorname{tr} C_p(A_1 + rI_q) \in W_{p,q}(A + rI)$. It follows that $E_1(\mu_1, \dots, \mu_q) \in \mathbf{R}$. This proves that $W_{1,q}(A) \subseteq \mathbf{R}$, and hence $A \in \mathscr{H}_n$ (see, for instance, Marcus and Sandy [17, Theorem 2]). ■

LEMMA 4. *Let p, q, n be integers such that $2 \leqslant p \leqslant q \leqslant n$. If a linear operator $T: \mathscr{S} \to \mathscr{S}$ satisfies*

$$W_{p,q}(T(A)) = W_{p,q}(A) \qquad \textit{for all} \quad A \in \mathscr{S},$$

where $\mathscr{S} = \mathscr{H}_n$ or $\mathbf{C}_{n\times n}$, then T is nonsingular.

Proof. We consider the case $\mathscr{S} = \mathscr{H}_n$ first. Suppose that there exists a nonzero matrix $A \in \mathscr{H}_n$ satisfying $T(A) = 0$. The desired contradiction will be obtained if we can find some $B \in \mathcal{H}_n$ satisfying $W_{p,q}(B) = \{0\}$ and $W_{p,q}(A+B) \neq \{0\}$; for then, we have

$$\{0\} \neq W_{p,q}(A+B) = W_{p,q}(T(A)+T(B)) = W_{p,q}(B) = \{0\}.$$

Write $A = UDU^*$, where $U \in \mathcal{U}_n$ and $D = \operatorname{diag}(\lambda_1,\ldots,\lambda_k,0,\ldots,0)$ such that $\lambda_1,\ldots,\lambda_k$ are all the nonzero eigenvalues of A, that is, $\operatorname{rank} A = k$. If $k \leqslant p$, take $B = UCU^*$, where $C = O_k \oplus I_{p-k} \oplus O_{n-p}$. Then $W_{p,q}(B) = \{0\}$, as $\operatorname{rank} B < p$. Further, since $J_q U^*(A+B)U = \operatorname{diag}(\lambda_1,\ldots,\lambda_k) \oplus I_{p-k} \oplus O_{n-p}$, we have $\prod_{i=1}^{k} \lambda_i \in W_{p,q}(A+B)$ and hence $W_{p,q}(A+B) \neq \{0\}$.

If $k > p$, let $B = U[\operatorname{diag}(\mu,0,\ldots,0)]U^*$, where μ is some real number to be chosen. Since $\operatorname{rank} B \leqslant 1 < p$, certainly $W_{p,q}(B) = \{0\}$. Note that $E_p(\lambda_1+\mu,\lambda_2,\ldots,\lambda_l) \in W_{p,q}(A+B)$, where $l = \min\{k,q\} \geqslant p$. If $l = p$, we can take $\mu = 0$ so that $E_p(\lambda_1+\mu,\lambda_2,\ldots,\lambda_l) \neq 0$, and we are done. If $l > p$, then

$$E_p(\lambda_1+\mu,\lambda_2,\ldots,\lambda_l) = E_p(\lambda_2,\ldots,\lambda_l) + (\lambda_1+\mu)E_{p-1}(\lambda_2,\ldots,\lambda_l).$$

In view of Lemma 1, there exists a real number μ such that $E_p(\lambda_1+\mu, \lambda_2,\ldots,\lambda_l) \neq 0$, and the result follows.

Now we come to the case $\mathcal{S} = \mathbf{C}_{n\times n}$. For $q = n$, the result is known (see [16, Lemma 3.1]). Hereafter we assume $q < n$. Let $A \in \mathbf{C}_{n\times n}$ such that $T(A) = 0$. Then for any $r \in \mathbf{R}$, we have

$$W_{p,q}(A+rI) = W_{p,q}(T(A)+T(rI))$$

$$= W_{p,q}(T(rI)) = W_{p,q}(rI) \subseteq \mathbf{R}.$$

By Lemma 3, $A \in \mathcal{H}_n$. We can now follow the same argument as in the hermitian case to conclude that $A = 0$. Therefore T is nonsingular. ∎

LEMMA 5. *Let p,q,n be integers such that $2 \leqslant p \leqslant q < n$. If a linear operator $T: \mathcal{S} \to \mathcal{S}$ satisfies*

$$W_{p,q}(T(A)) = W_{p,q}(A) \qquad \textit{for all} \quad A \in \mathcal{S},$$

where $\mathcal{S} = \mathcal{H}_n$ or $\mathbf{C}_{n\times n}$, then there exists a complex (real if $\mathcal{S} = \mathcal{H}_n$) number ξ with $\xi^p = 1$ such that $T(I) = \xi I$.

Proof. Here we give the proof for the complex case. For the hermitian case a much simpler proof along the same lines can be given.

Denote by $\mathscr{E}$ the set

$$\left\{A \in \mathbf{C}_{n\times n} : W_{p,q}(A) = W_{p,q}(I) = \left\{\binom{q}{p}\right\}\right\}.$$

Since T preserves the (p,q)-numerical range and is nonsingular by Lemma 4, we have $T(\mathscr{E}) = \mathscr{E}$. The conclusion will follow if we can show that the set $\{\xi I : \xi^p = 1\}$ is the set of isolated points of $\mathscr{E}$.

First, we show that if $A \in \mathscr{E}$ has all eigenvalues equal, then A must be a scalar matrix. Assume that the contrary holds. Replacing A by a suitable unitary transform, we may suppose that A is in lower triangular form with some nonzero off-diagonal entries and all diagonal entries being equal to λ. Choose a nonzero off-diagonal entry a_{ij} of A with the property that $a_{ik} = 0$ for $j < k < i$ and $a_{tj} = 0$ for $j < t < i$. Denote by $A[j,i]$ the principal submatrix of A formed by its j, i rows and columns. For any β in $W(A[j,i])$, the numerical range of $A[j,i]$, which in this case is a circular disk, we can find $V \in \mathscr{U}_2$ such that the (1,1) entry of $VA[j,i]V^*$ is β. Let $U \in \mathscr{U}_n$ be obtained from the identity matrix by replacing its (j,j), (j,i), (i,j), and (i,i) entries with the (1,1), (1,2), (2,1), and (2,2) entries of V respectively. Then UAU^* is almost a lower triangular matrix except that its (j,i) entry may be nonzero; moreover, its (j,j) entry equals β, (i,i) entry equals $2\lambda - \beta$, and all other diagonal entries equal λ. It follows that β can be chosen so that

$$\binom{q}{p} \neq E_p\Big(\beta, \underbrace{\lambda, \ldots, \lambda}_{q-1 \text{ times}}\Big) \in W_{p,q}(A).$$

This contradicts our assumption that $A \in \mathscr{E}$.

Evidently, for any $\xi \in \mathbf{C}$, $\xi I \in \mathscr{E}$ if and only if $\xi^p = 1$. Let f be the complex-valued function on $\mathbf{C}_{n\times n}$ defined by

$$f(A) = \prod_{1 \leq i_1 < \cdots < i_{q-1} \leq n} E_{p-1}\left(\lambda_{i_1}, \ldots, \lambda_{i_{q-1}}\right),$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. We are going to show that for any $A \in \mathscr{E}$, $f(A) \neq 0$ if and only if A is a scalar matrix. If $A = \xi I$ ($\xi^p = 1$) then clearly $f(A) \neq 0$. If $A \in \mathscr{E}$ is not a scalar matrix, then as shown above A has at least two distinct eigenvalues, say $\lambda_1 \neq \lambda_2$. In view of

$$\binom{q}{p} = E_p(\lambda_i, \lambda_3, \ldots, \lambda_{q+1})$$

$$= E_p(\lambda_3, \ldots, \lambda_{q+1}) + \lambda_i E_{p-1}(\lambda_3, \ldots, \lambda_{q+1})$$

for $i = 1,2$, where we set by convention $E_p(\lambda_3, \ldots, \lambda_{q+1}) = 0$ if $p = q$, it follows that $E_{p-1}(\lambda_3, \ldots, \lambda_{q+1}) = 0$. Hence $f(A) = 0$.

Finally, we prove that the set of isolated points of $\mathscr{E}$ consists of scalar matrices. If $A \in \mathscr{E}$ is not a scalar matrix, then $\{UAU^* : U \in \mathscr{U}_n\}$ is an infinite pathwise-connected subset of $\mathscr{E}$ containing A. Hence A is an accumulation point of $\mathscr{E}$. Conversely, suppose that A is an accumulation point of $\mathscr{E}$. Then there exists a sequence $(A_k)_{k \in \mathbb{N}}$ with distinct terms in $\mathscr{E}$ converging to A. Since $\mathscr{E}$ contains only finitely many scalar matrices, we may assume that each A_k is a nonscalar matrix, and hence satisfies $f(A_k) = 0$. By the continuity of f, we also have $f(A) = 0$, and so A is a nonscalar matrix. The proof is complete. ∎

3. PROOF OF THE MAIN RESULTS

Proof of Theorem 1. The "if" part of the result is obvious.

"Only if": By Lemma 4, T is nonsingular. Let $A \in \mathscr{H}_n$ be of rank 1. Then for any $B \in \mathscr{H}_n$ and $U \in \mathscr{U}_n$, we have

$$\operatorname{tr} C_p\left(J_q U\left[xT(A) + B\right]U^*\right) \in W_{p,q}(xT(A) + B) = W_{p,q}\left(xA + T^{-1}(B)\right).$$

Hence by Lemma 2, $\operatorname{rank} T(A) = 1$. Thus T is a rank-one preserver. We can now apply Lemma 2 in Johnson and Pierce [6] to conclude that there exist a nonsingular matrix $X \in \mathbb{C}_{n \times n}$ and an $\xi = 1$ or -1 such that

$$T(A) = \xi XA^+X^* \qquad \text{for all} \quad A \in \mathscr{H}_n^o,$$

where A^+ denotes A or A^t. We will show that in all cases T is of the required form.

In case $2 \leqslant p = q = n$, i.e., when T preserves the determinant function, we have,

$$1 = \det I = \det(\xi XX^*) = \xi^n |\det X|^2,$$

and hence $|\det X| = 1$ and $\xi^n = 1$.

In case $2 \leqslant p \leqslant q < n$, by Lemma 5, we have $\xi XX^* = T(I) = \varepsilon I$ for some $\varepsilon \in \mathbb{R}$ with $\varepsilon^p = 1$. Hence $\xi = \varepsilon$ satisfies $\xi^p = 1$, and X is unitary.

For the remaining case $2 < p < q = n$, i.e., when T preserves the pth elementary symmetric function of the eigenvalues of a matrix, we proceed as follows. [Only the case when $T(A) = \xi XAX^*$ for all $A \in \mathscr{H}_n$ will be considered.] Write $X = W\operatorname{diag}(s_1, \ldots, s_n)V$, where $W, V \in \mathscr{U}_n$ and $s_1, \ldots, s_n$ are the singular values of X arranged in nonincreasing order. Take $A = V^*(I_p \oplus O_{n-p})V$. We have

$$T(A) = \xi W \operatorname{diag}\left(s_1^2, \ldots, s_p^2, 0, \ldots, 0\right) W^*.$$

Then the values of E_p at A and $T(A)$ are respectively 1 and $\xi^p(\prod_{i=1}^p s_i^2)$. (Recall that $\xi = \pm 1$.) Hence $\xi^p = 1$ and $\prod_{i=1}^p s_i = 1$. Similarly, by choosing $A = V^*(O_{n-p} \oplus I_p)V$ we can show that $\prod_{i=n-p+1}^n s_i = 1$. But $s_1 \geqslant \cdots \geqslant s_n > 0$; it follows that $s_1 = \cdots = s_n = 1$. Thus X is unitary and $\xi^p = 1$. ∎

Proof of Theorem 2. We need to consider only the "only if" part. By Lemma 5, there exists a complex pth root of unity ξ such that $T(I) = \xi I$. Replacing T by $\bar{\xi}T$, we may assume that $T(I) = I$. Then it follows readily from Lemma 3 that $T(\mathscr{H}_n) \subseteq \mathscr{H}_n$. In view of Theorem 1 and the fact that $T(I) = I$, there exists $U \in \mathscr{U}_n$ such that

$$T(A) = UA^+U^* \qquad \text{for all} \quad A \in \mathscr{H}_n,$$

where A^+ denotes A or A^t. But T is linear (over $\mathbb{C}$) and $\mathbb{C}_{n\times n} = \mathscr{H}_n + i\mathscr{H}_n$. So our theorem follows readily. ∎

4. A VARIATION OF THE PROBLEM

Let p, q, m, n be integers such that $1 \leqslant p \leqslant q \leqslant \min\{m, n\}$. For any $A \in \mathbb{C}_{m\times n}$, one may consider the set

$$D_{p,q}(A) = \left\{\operatorname{tr} C_p\left(K_q UAV\right) : U \in \mathscr{U}_m \text{ and } V \in \mathscr{U}_n\right\},$$

where K_q is the $n\times m$ matrix with 1's in the $(1,1),(2,2),\dots,(q,q)$ entries and 0's elsewhere. In the case when $m=n$, $W_{p,q}(A)$ is clearly contained in $D_{p,q}(A)$. It is natural to ask what kind of linear operators on $\mathbf{C}_{m\times n}$ preserve $D_{p,q}(\cdot)$. Using some known related results, we are now ready to answer this question.

In [24, Theorem 1] Thompson has described the set $D_{p,q}(A)$ for square matrices A. His results can be extended readily to rectangular matrices as follows: if $A\in\mathbf{C}_{m\times n}$ has singular values $s_1(A)\geqslant\cdots\geqslant s_t(A)$, where $t=\min\{m,n\}$, then, except for the case when p,q,m,n are all equal, $D_{p,q}(A)$ is a circular disk centered at the origin in the complex plane with radius $E_p(s_1(A),\dots,s_q(A))$; for the exceptional case, $D_{p,q}(A)$ is a circle with the origin as center and radius $|\det A|$. An easy consequence of this result is the following: A linear operator $T:\mathbf{C}_{m\times n}\to\mathbf{C}_{m\times n}$ satisfies

$$D_{p,q}(T(A))=D_{p,q}(A)\qquad\text{for all}\quad A\in\mathbf{C}_{m\times n}$$

if and only if for all $A\in\mathbf{C}_{m\times n}$,

$$E_p\big(s_1(T(A)),\dots,s_q(T(A))\big)=E_p\big(s_1(A),\dots,s_q(A)\big).$$

On the other hand, Marcus and Gordon [14, Theorem 3] and Grone [4, Theorem 4] have characterized those linear operators on $\mathbf{C}_{m\times n}$ which preserve the pth elementary symmetric function of the singular values of matrices. Borrowing their results (and proofs), we can readily state:

THEOREM 3. *Let p,q,m,n be integers such that $1\leqslant p\leqslant q\leqslant\min\{m,n\}$. A linear operator $T:\mathbf{C}_{m\times n}\to\mathbf{C}_{m\times n}$ satisfies*

$$D_{p,q}(T(A))=D_{p,q}(A)\qquad\textit{for all}\quad A\in\mathbf{C}_{m\times n}$$

if and only if there exist $U\in\mathbf{C}_{m\times m}$ and $V\in\mathbf{C}_{n\times n}$ such that

$$T(A)=UAV\qquad\textit{for all}\quad A\in\mathbf{C}_{m\times n},$$

or possibly

$$T(A)=UA^tV\quad\textit{for all } A\in\mathbf{C}_{m\times n}\qquad\textit{when}\quad m=n,$$

where U and V satisfy

(i) *if* $p < \min\{m, n\}$, *then U and V are unitary;*
(ii) *if* $p = m < n$, *then* $|\det U| = 1$ *and V is unitary;*
(iii) *if* $p = n < m$, *then U is unitary and* $|\det V| = 1$;
(iv) *if* $p = m = n$, *then* $|\det(UV)| = 1$.

5. SOME REMARKS AND QUESTIONS

Recall that the kth derivation of a matrix A on the qth Grassmann space over $\mathbb{C}^n$, denoted by $D_k(A)$, is given by the formula

$$C_q(I + tA) = \sum_{r=0}^{q} t^r D_r(A).$$

As indicated by M. Marcus, the set $W_{p,q}(A)$ can also be regarded as the decomposable numerical range of $D_p(A)$, i.e.,

$$W_{p,q}(A) = \Big\{\big\langle D_p(A)x^\wedge, x^\wedge \big\rangle : x^\wedge \text{ is a decomposable unit vector in the } q\text{th Grassmann space of } \mathbb{C}^n\Big\}.$$

(cf. Marcus and Sandy [18, Section 2].)

Before we end, we say a few words about possible research along the direction of this work.

One may generalize the definition of the set $D_{p,q}(A)$ by replacing the matrix K_q with an arbitrary matrix C in $\mathbb{C}_{m\times n}$, and ask the corresponding linear-preserver problem. It is not difficult to extend the result of Theorem 3 (with q replaced by $\operatorname{rank} C$). Indeed, the corresponding result for the case when $p = 1$ can be found in Li and Tsing [11]. Similarly, the definition of the set $W_{p,q}(A)$ can also be generalized. But extending the results of Theorems 1 and 2 is a much harder problem. In the case when C is hermitian, if $p = 1$ the corresponding result can be found in Li and Tsing [10]; if $p \geqslant 3$, the method developed in this paper can also be applied. However, when $\mathcal{S} = \mathbb{C}_{n\times n}$ and C is an arbitrary matrix in $\mathbb{C}_{n\times n}$, the problem remains unsolved.

Finally, one may further generalize the above definitions by taking an arbitrary induced matrix instead of taking the pth compound matrix. Again, the result of Theorem 3 can be extended without difficulty. However, not much can be said about the other cases.

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